Algorithmic Fair Allocation of Indivisible Items: A Survey and New Questions

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The theory of algorithmic fair allocation is at the center of multi-agent systems and economics in recent decades due to its industrial and social importance. At a high level, the problem is to assign a set of items that are either goods or chores to a set of agents so that every agent is happy with what she obtains. In this survey, we focus on indivisible items, for which exact fairness as measured by envy-freeness and proportionality cannot be guaranteed. One main theme in the recent research agenda is designing algorithms that approximately achieve fairness criteria. We aim at presenting a comprehensive survey of recent progressthrough the prism of algorithms, highlighting the ways to relax fairness notions and common techniques to design algorithms, as well as the most interesting questions for future research.

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1. INTRODUCTION

While fair allocation is an age-old problem and the widely known Divide-and-Choose algorithm can be traced back to the Bible, modern research on fair allocation is regarded to be initiated by Steinhaus at a meeting of the Econometric Society in Washington D.C. in 1947 (Steinhaus, 1948). Since then, a large body of workin economics and mathematics has been directed towards understanding the theory of allocating resources among agents in a fair manner (Moulin, 2003). The recent focus on *indivisible* items is motivated, in part, by the applications that inherently entail allocation of items that cannot be fractionally allocated, such as assigning computational resources in a cloud computing environment and courses

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to teachers in a school. In the last decade, computer science has offered a fresh and practical angle to the research agenda – algorithmic fair allocation. In addition to designing algorithms, computer science has brought many more ideas, such as computational and communication complexity, and informational assumptions, which do not align with the main theme of the current survey. Interested readers can refer to the surveys by Walsh (2020) and Aziz (2020a) for detailed discussion. Fair allocation algorithms have been implemented in the real world; for instance, Course Match is employed for course allocation at the Wharton School in the University of Pennsylvania, and the websites Spliddit (`spliddit.org`) and Fair Outcomes (`fairoutcomes.com`) provide online access to fair allocation algorithms.

Although this survey mainly focuses on indivisible items, the study of fair allocation was classically centered around allocating a divisible resource, which is also known as the *cake-cutting problem* (Brams and Taylor, 1996; Robertson and Webb, 1998). Fairness is mostly captured by *envy-freeness* and *proportionality* in the literature. An envy-free allocation (which is also proportional) of a divisible cake always exists and can be found in bounded steps (Aziz and Mackenzie, 2016). Moreover, a competitive equilibrium from equal incomes guarantees envy-freeness and Pareto optimality simultaneously (Varian, 1973). A recent line of research extends the study to chores, such as the computation of envy-free allocations (Dehghani et al., 2018) and competitive equilibria (Boodaghians et al., 2021; Chaudhury et al., 2021a). Unlike divisible items, when items are indivisible, absolutely fair allocations rarely exist. For example, when allocating a single item to two agents, no allocation is envy-free or proportional. Accordingly, an extensively studied subject is to investigate the extent to which these fairness notions or their relaxations can be approximately satisfied.

There are several surveys highlighting different perspectives of fair allocation theory. Moulin (2018) reviewed the theory through the prism of economics. Aleksandrov and Walsh (2020) and Suksompong (2021) respectively focused on the online and constrained settings. Lang and Rothe (2016), Walsh (2020) and Aziz (2020a) reviewed the problem in the perspective of broad computer science. Instead, the angle of the current survey is algorithmic and the focus is particularly on the introduction of common techniques to design (approximation) algorithms. Moreover, we will discuss more sophisticated settings introduced in the last couple of years that uncovered new challenges and open problems in the field of fair allocation.

Roadmap. In the remaining of the survey, we define the model of fair allocation in Section 2 and introduce the widely adopted solution concepts in Sections 3 and 4. In Section 5, we review the commonly used techniques to design fair allocation algorithms. In Section 6, we introduce more sophisticated settings that have been proposed recently. Finally, we discuss two more propertiesthat may be desirable to be satisfied together with fairness, efficiency and truthfulness, in Section 7.

2. MODEL

In a fair allocation instance, we allocate a set of m indivisible items $M = \{1, \dots, m\}$ to a group of n agents $N = \{1, \dots, n\}$. An allocation is represented by an n-partition $\mathbf{X} = (X_1, \dots, X_n)$ of M, where $X_i \subseteq M$ is the bundle allocated to agent i. It is required that each item is allocated to exactly one agent, i.e., $X_i \cap X_j = \emptyset$ for all

$i \neq j$ and $\cup_{i\in N} X_i = M$. If $\cup_{i\in N} X_i \neq M$, $\mathbf{X}$ is a *partial* allocation. We sometimes consider fractional allocations, denoted by $\mathbf{x} = (x_{ie})_{i\in N, e\in M}$, where $0 \leq x_{ie} \leq 1$ denotes the fraction of item e allocated to agent i, and $\sum_{i\in N} x_{ie} = 1$ for all $e \in M$. Each agent i has a valuation function $v_i : 2^M \to \mathbb{R}$ that assigns a value to each bundle of items. When $v_i(S) \geq 0$ for all $i \in N$ and $S \subseteq M$, the items are *goods*; when $v_i(S) \leq 0$ for all i and S, the items are *chores*. For ease of exposition, we mainly discuss the case when the valuations are additive and leave the discussion on more general valuations to Section 6. That is, for any $i \in N$ and $S \subseteq M$, we have $v_i(S) = \sum_{e\in S} v_i(\{e\})$. When there is no confusion, we use $v_i(e)$ to denote $v_i(\{e\})$. Further, for any $S \subseteq M$ and $e \in M$, we use $S + e$ and $S - e$ to denote $S \cup \{e\}$ and $S \setminus \{e\}$, respectively. Let $\mathcal{I} = (N, M, \mathbf{v})$ be a fair allocation instance where $\mathbf{v} = (v_1, \ldots, v_n)$. When all agents agree on the same ordering of all items in values (i.e., $v_i(1) \geq \cdots \geq v_i(m)$ for all $i \in N$), the instance is called *identical ordering* (IDO).

Before the extensive study of fairness, efficiency was at the centre of the theory of resource allocation. The *utilitarian welfare* of an allocation $\mathbf{X}$ is $\sum_{i\in N} v_i(X_i)$, by maximizing which the total happiness of the agents is maximized. The *egalitarian welfare* is $\min_{i\in N}\{v_i(X_i)\}$, by maximizing which the smallest happiness is maximized. A compromise between utilitarian and egalitarian welfare is *Nash welfare*, i.e., $\Pi_{i\in N} v_i(X_i)$. We say an allocation $\mathbf{X}$ *Pareto dominates* another allocation $\mathbf{X}'$ if $v_i(X_i) \geq v_i(X_i')$ for all $i \in N$ and $v_i(X_i) > v_i(X_i')$ for some i. An allocation is *Pareto optimal* (PO) if it is not Pareto dominated by any other allocation.

Naturally, the fairness of an allocation can be evaluated by its egalitarian welfare, as in the Santa Claus problem (Bansal and Sviridenko, 2006) and the load balancing problem (Lenstra et al., 1990). However, in practice, since agents may have heterogeneous valuations, the max-min objective is not enough to satisfy all of them. Thus, various notions were proposed to characterize the fairness of allocations, including envy-freeness (EF) (Foley, 1966), proportionality (PROP) (Steinhaus, 1948) and equitability (EQ) (Dubins and Spanier, 1961). The relationships among these notions are discussed by Amanatidis et al. (2018), Sun et al. (2021) and Chakraborty et al. (2021). Given the vast literature on different fairness notions, this survey only focuses on two of the most widely studied, namely EF and PROP.

3. ENVY-FREENESS

We first consider envy-freeness, the study of which dates back to Foley (1966) and Tinbergen (1930), and its relaxations.

Definition 1 *EF*. For the allocation of items (goods or chores), an allocation $\mathbf{X}$ is envy-free (EF) if for any two agents $i, j \in N$, we have $v_i(X_i) \geq v_i(X_j)$.

The problem of checking whether a given instance admits an EF allocation is NP-complete even for $\{0, 1\}$- or $\{0, -1\}$-valued instances (Aziz et al., 2015; Bhaskar et al., 2021). Moreover, the example of allocating a single item between two agents defies any bounded multiplicative approximation of EF, and thus researchers turn their attention to additive approximations. Two of the most popular ones are envy-free up to one item (EF1) and envy-free up to any item (EFX).

The notion of EF1 was first studied for the allocation of goods by Lipton et al. (2004), which allows an agent to envy another agent but requires that the envy can be eliminated by removing an item from the envied agent's bundle. This notion naturally extends to chores by removing an item from the envious agent's bundle. For both goods and chores, EF1 allocations always exist and can be efficiently computed by the Round Robin algorithm; see Section 5.

Definition 2 *α-EF1*. For any $\alpha \geq 0$, an allocation $\mathbf{X}$ is α-approximate envy-free up to one item (α-EF1) if for any $i, j \in N$, there exists $e \in X_i \cup X_j$ such that $v_i(X_i - e) \geq \alpha \cdot v_i(X_j - e)$. When $\alpha = 1$, the allocation is EF1.

As with EF1, the EFX relaxation was proposed for the allocation of goods, by Caragiannis et al. (2019b). Informally speaking, the notion of EFX strengthens the fairness by requiring that the envy between two agents can be eliminated by removing *any* item owned by these two agents.

Definition 3 *α-EFX*. For any $\alpha \geq 0$, an allocation $\mathbf{X}$ for goods (resp. chores) is α-approximate envy-free up to any item (α-EFX) if for any $i, j \in N$ and any $e \in X_j$ (resp. $e \in X_i$), $v_i(X_i) \geq \alpha \cdot v_i(X_j - e)$ (resp. $v_i(X_i - e) \geq \alpha \cdot v_i(X_j)$). When $\alpha = 1$, the allocation is EFX.

Unlike the case of EF1 allocations, the existence of EFX allocations remains unknown. For the case of goods, it was shown by Plaut and Roughgarden (2020) that EFX allocations exist in some special cases: (1) identical (combinatorial) valuations, (2) IDO additive valuations, and (3) $n = 2$. Chaudhury et al. (2020) and Amanatidis et al. (2021a) further extended the existence of EFX allocations to the cases when (4) $n = 3$, and (5) bi-valued valuations. In contrast to the case of goods, the chores counterpart is much less well studied. EFX allocations for chores are known to exist only for a few special cases, e.g., IDO instances (Li et al., 2022) and leveled preference instances (Gafni et al., 2021). The existence of EFX allocations for chores remains unknown even for $n = 3$ agents or bi-valued instance.

Open Problem 1. *Do EFX allocations always exist (for both goods and chores)?*

While the existence of EFX allocations remains unknown for the general cases, there are fruitful results regarding EFX partial allocations (where unallocated items are assumed to be donated to a *charity*) and approximation of EFX allocations.

Since allocating nothing to the agents is trivially EFX, researchers are interested in finding EFX partial allocations with high efficiency. Caragiannis et al. (2019a) showed that there exists an EFX partial allocation achieving half of the maximum Nash welfare. Chaudhury et al. (2021d) proposed a pseudo-polynomial time algorithm that computes an EFX partial allocation with at most $n - 1$ unallocated items under which no agent envies the charity. This result was improved by Berger et al. (2021), who showed that there is an EFX allocation with at most a single unallocated item for $n = 4$, and $n - 2$ unallocated items for $n \geq 5$.

There are also results that aim at computing approximately EFX allocations. Plaut and Roughgarden (2020) showed that every instance (even with subadditive valuations) admits a 0.5-EFX allocation. The approximation ratio was improved to 0.618 under additive valuations by a polynomial time algorithm proposed by Amanatidis et al. (2020). Chaudhury et al. (2021b) proposed a polynomial time algorithm that computes a $(1-\epsilon)$-EFX allocation with $o(n)$ unallocated items and high Nash welfare. For the allocation of chores, only an $O(n^2)$ approximation of EFX is known to exist (Zhou and Wu, 2021).

4. PROPORTIONALITY AND MAXIMIN SHARE

Proportionality (PROP) was proposed by Steinhaus (1948), and is the most widely studied threshold-based solution concept. PROP is weaker than EF under additive valuations.

Definition 4 *PROP*. An allocation $\mathbf{X}$ is proportional (PROP) if for every agent $i \in N$, we have $v_i(X_i) \geq \mathsf{PROP}_i$, where $\mathsf{PROP}_i = (1/n) \cdot v_i(M)$.

For divisible goods and normalized valuations, the items can be allocated such that every agent has value at least $1/n$, which is not true for indivisible items. Hill (1987) studied the worst case guarantee that an agent can have as a function of n and $\max_{i \in N, e \in M}\{v_i(e)\}$. With two agents, the chores version is equivalent to the goods one; but with three or more agents, the equivalence is far from clear, and may not hold. One drawback of this guarantee is that the value of the function decreases quickly and goes to 0 as $\max_{i,e}\{v_i(e)\}$ becomes large.

Open Problem 2. *For chores, what is the worst case guarantee that an agent has as a function of n and the values of the agents?*

Maximin Share Fairness. Besides the worst case guarantee studied by Hill (1987), one popular relaxation of PROP is maximin share fairness, motivated by the following imaginary experiment. If agent i is the mediator and divides all items into n bundles, the best way to approximate PROP for i is to maximize the smallest bundle according to v_i. Formally, define the maximin share (MMS) of i as

$$\mathsf{MMS}_i(M, n) = \max_{\mathbf{X} \in \Pi_n(M)} \min_j \{v_i(X_j)\},$$

where $\Pi_n(M)$ denotes the set of all n-partitions of M. When M and n are clear from the context, we write MMS_i for short. Note that $\mathsf{MMS}_i \leq \mathsf{PROP}_i$, and the computation of MMS_i is NP-complete.

Definition 5 *α-MMS*. For any $\alpha \geq 0$, an allocation $\mathbf{X}$ is α-approximate maximin share fair (α-MMS) if for any $i \in N$, we have $v_i(X_i) \geq \alpha \cdot \mathsf{MMS}_i$. When $\alpha = 1$, the allocation is MMS.

Note that the approximation ratio $\alpha \leq 1$ for goods and $\alpha \geq 1$ for chores. The definition of MMS fairness was first introduced by Budish (2011), based on the concept of (Moulin, 1990). Unfortunately, it is shown that there exist instances for which no allocation can ensure MMS_i value for every agent for the case of goods (Kurokawa et al., 2018) and chores (Aziz et al., 2017b). The best known approximation results are $(3/4 + 1/(12n))$-MMS for goods (Garg and Taki, 2021) and 11/9-MMS for chores (Huang and Lu, 2021). The best known negative results are that $\alpha \leq 39/40$ for goods and for $\alpha \geq 44/43$ for chores by Feige et al. (2021).

Open Problem 3. *What are the best possible approximation ratios of MMS allocations (for both goods and chores)?*

More Solution Concepts. Motivated by the definition of MMS, Caragiannis et al. (2019b) proposed *pairwise MMS* (PMMS) and Barman et al. (2018a) proposed *groupwise MMS* (GMMS). Informally, PMMS is similar to MMS, but instead requires that the allocation is MMS for the instance induced by any two agents. GMMS generalizes both MMS and PMMS and requires that the allocation is MMS for the instance induced by any subset of agents. We refer the readers to, e.g., (Caragiannis et al., 2019b; Barman et al., 2018a; Amanatidis et al., 2020), for more detailed discussions, and we summarize the main open problem as follows.

Open Problem 4. *Do PMMS allocations always exists? What is the best possible approximation of GMMS?*

Finally, similar to EF1 and EFX, we can relax PROP to PROP1 and PROPX. It is known that a PROP1 allocation always exists and can be found in polynomial time when the items are goods (Conitzer et al., 2017; Barman and Krishnamurthy, 2019), chores (Brânzei and Sandomirskiy, 2019) or mixture of goods and chores (Aziz et al., 2020b). Regarding PROPX, when items are goods, PROPX allocations may not exist (Moulin, 2018; Aziz et al., 2020b). However, when items are chores, PROPX allocations exist and can be found efficiently (Moulin, 2018; Li et al., 2022). Recently, Baklanov et al. (2021) further proposed PROPm for goods that sits between PROP1 and PROPX, and is guaranteed to exist.

5. ALGORITHMS AND COMMON TECHNIQUES

In this section, we introduce the techniques to design fair allocation algorithms. Due to the vast literature, we choose some of the most commonly used and powerful ones that are also the basis of more complicated algorithms.

5.1 Divide-and-Choose

Divide-and-Choose is one of the most classic allocation algorithms. The algorithm is very useful and intuitive when there are only two agents. The idea is to let the first agent partition the items into two bundles and the other agent choose her preferred bundle. The remaining bundle is allocated to the first agent, and thus her best strategy is to maximize the value of the smaller bundle, i.e.,

$$(X_1, X_2) \in \arg \max_{(S_1,S_2)\in\Pi_2(M)} \min\{v_1(S_1), v_1(S_2)\}.$$

Plaut and Roughgarden (2020) proved that (X_1, X_2) is always MMS and EFX to the first agent if we break ties by maximizing the size of the smaller bundle in (X_1, X_2), which they term the leximin++ allocation. This result holds for any number of agents, which implies the existence of EFX allocations for the case of identical valuations. Since the second agent obtains her preferred bundle in (X_1, X_2), the allocation is EF to her. Therefore, with two agents, Divide-and-Choose algorithm returns an allocation that is MMS and EFX.

5.2 Adjusted-Winner

Adjusted-Winner is another widely used algorithm for the two-agent case (Brams and Taylor, 1996). The idea is to sort the items according to the ratios between the utilities that they yield for the two agents, i.e.,

$$\frac{v_1(1)}{v_2(1)} \geq \frac{v_1(2)}{v_2(2)} \geq \cdots \geq \frac{v_1(m)}{v_2(m)},$$

and let agent 1 choose a minimal set of consecutive items for which she is EF1 starting from left (the remaining items are given to agent 2). The advantage is that it ensures high social welfare between two agents (Bei et al., 2021c). This allocation is EF1 but not necessarily MMS or EFX.

5.3 Sequential Allocation and Round-Robin

A general class of algorithms that are also suitable for a distributed implementation is that of sequential-picking allocation (Brams and Taylor, 2000), which was formally studied in a general and systematic way by Bouveret and Lang (2011). Under these methods, agents have a sequence of turns to pick their most preferred item that is still available. A popular sequence protocol is the *Round-Robin*, where the picking sequence repeats the pattern $1, \ldots, n$. The Round-Robin algorithm produces allocations that are EF1 (but not necessarily EFX) for both goods and chores[1], but not for mixtures of them. For this, Aziz et al. (2020b) proposed the double Round-Robin method that computes EF1 allocations for mixture of goods and chores. Amanatidis et al. (2016) and Aziz et al. (2020a) designed more involved picking sequences to approximate MMS fairness, for goods and chores respectively.

5.4 Envy-cycle Elimination

The Envy-cycle Elimination algorithm is inherently a greedy algorithm, where in each round a new item is assigned to the agent who is at a disadvantage for goods or advantage for chores (Lipton et al., 2004). The main technique of the algorithm is to ensure the existence of an agent that is not envied (for goods) or not envious (for chores) by trading items among agents. We use goods as an illustration. The algorithm is based on an *envy graph*, where the nodes correspond to agents and there is an edge from agent i to agent j if i is envious of j's bundle. The algorithm works by assigning, at each step, an unassigned item to an agent who is not envied by any other agents, i.e., a node with in-degree 0 in the envy graph. If no such agent exists, the graph must contain a directed cycle. Then the cycle can be resolved by exchanging the bundles of items along the cycle, i.e., an agent in the cycle gets the bundle of the agent she points to. The algorithm terminates when all items are allocated and outputs an EF1 allocation for arbitrary monotone combinatorial valuation functions (Lipton et al., 2004).

The algorithm and its adaptations are very widely studied, combining with which stronger fairness notions can also be satisfied. For example, the algorithm itself ensures EFX (Plaut and Roughgarden, 2020) and 2/3-MMS (Barman and Krishnamurthy, 2020) for IDO instances. With more involved preprocessing procedures, it can ensure 0.618-EFX, 0.553-GMMS, 0.667-PMMS and EF1 simultaneously (Amanatidis et al., 2020). For chores, it is shown in (Barman and Krishnamurthy, 2020)

[1]When items are only goods or only chores, there is a larger class of protocols ensuring EF1. This class of protocols uses a *recursively balanced* sequence in which at any point, the difference between the number of turns of any of two agents is at most 1.

that the returned allocation is 4/3-MMS. However, noted by Bhaskar et al. (2021), this allocation may not be EF1 if the cycle is resolved arbitrarily. Instead, they used the top-trading technique (in which each agent only points to the agent she envies the most) to preserve EF1. Later, Li et al. (2022) further showed that with this technique, the returned allocation is PROPX. We can also observe the shadow of the algorithm in more complicated techniques, such as the (group) champion graphs and rainbow cycle number (Chaudhury et al., 2020, 2021b) which enable stronger existence and approximation results of EFX.

5.5 Bag-filling Algorithms

Bag-filling Algorithms are particularly helpful for threshold-based notions of fairness like MMS. The idea is to maintain a bag and keep adding items to it until some agent thinks the bag is good enough (for goods) or about to be too bad to all agents (for chores). Then the bag is taken away by some satisfied agent and the algorithm repeats the procedure with the remaining items. The difficulty is to select a proper threshold for the bag so that the approximation for the agent who takes away a bag is good and there remains sufficiently many (or few) items for the remaining agents. With a more careful design and analysis, the approximation ratio can be improved to 2/3 (Garg et al., 2019) and further better than 3/4 (Garg and Taki, 2021) for goods, For chores the approximation ratio can be improved to 11/9 (Huang and Lu, 2021). There are several nice properties regarding MMS fairness (Amanatidis et al., 2017b; Garg et al., 2019), e.g., scale invariance and a reduction to IDO instances. Interestingly, the second property shows that any algorithm for approximating MMS allocations for IDO instances applies to general instances with the same approximation ratio preserved. Making use of these properties can significantly simplify the design of algorithms.

5.6 Rounding Fractional Solutions

Although competitive equilibria may not exist for indivisible items, we can first compute a market equilibrium by assuming the items are divisible and then carefully round the fractional allocation to an integral one (Barman and Krishnamurthy, 2019; Brânzei and Sandomirskiy, 2019; Garg et al., 2021a). This approach is especially helpful when efficiency is desired along with fairness, e.g., for the computation of EF1+PO or PROP1+PO allocations for goods (Barman et al., 2018b; Barman and Krishnamurthy, 2019), EF1+PO allocations for bi-valued chores (Garg et al., 2021b; Ebadian et al., 2021), PROP1+PO (Aziz et al., 2020b) and approximately MMS+PO allocations for mixture of goods and chores (Kulkarni et al., 2021).

5.7 Eating Algorithms

The Probabilistic-Serial (PS) algorithm of Bogomolnaia and Moulin (2001) is a randomized algorithm for allocating indivisible items in an ex-ante EF manner. Agents eat their most preferred items at a uniform rate and move on to the next item when the previous one is consumed. The probability share of an agent for an item is the fraction of the item eaten by the agent. In recent works (Freeman et al., 2020; Aziz, 2020b), researchers have sought allocation algorithms that simultaneously satisfy ex-ante EF and ex-post EF1 for the allocation of indivisible items that are goods or chores. In particular, the PS-lottery method was proposed that provides an explicit lottery over a set of EF1 allocations. Aziz and Brandl (2020) presented an eating algorithm that is suitable for any type of feasibility constraint and allocation problem with ordinal preferences.

6. MORE SOPHISTICATED SETTINGS

The past several years have witnessed the emergence of more sophisticated settings that brought new challenges to the design of fair allocation algorithms, including the mixture of goods and chores, weighted agents, partial information and general valuations. In the following, we review their models, as well as the corresponding results and open problems.

6.1 Mixture of Goods and Chores

The general case when items are mixture of goods and chores has recently been studied in (Bogomolnaia et al., 2017, 2019; Aziz et al., 2020b, 2022). This model is particularly interesting because it includes the typical setting when the valuations are not monotone. Aziz et al. (2022) proved that a double Round Robin algorithm is able to compute an EF1 allocation for any number of agents, and a generalized adjusted winner algorithm can find an EF1 and PO allocation for two agents. A natural open question is whether PO and EF1 allocations exist for arbitrary number of agents. Recently, Aziz et al. (2020b) and Kulkarni et al. (2021) designed algorithms that compute PROP1+PO or approximately MMS+PO allocations, respectively. More generally, it is an intriguing future research direction to study the fair allocation problem under other non-monotonic valuations.

6.2 Asymmetric Agents

For most of the aforementioned research works, the agents are assumed to be symmetric in the sense of taking the same share in the system. Motivated by real-world scenarios where people in leadership positions take more responsibilities, some recent works studied the fair treatment of non-equals. The definitions of envy-freeness and maximin share fairness have been adapted to the weighted settings by Farhadi et al. (2019), Aziz et al. (2019a) and Chakraborty et al. (2020). Regarding goods, it is shown by Farhadi et al. (2019) that the best approximation ratio for weighted MMS is $\Theta(n)$ and by Chakraborty et al. (2020) that weighted EF1 allocations exist. Regarding chores, although weighted MMS was studied by Aziz et al. (2019a), the best approximation ratio and the existence of weighted EF1 allocations are still unknown. Novel fairness notions, such as AnyPrice share and l-out-of-d maximin share, were proposed and studied by Babaioff et al. (2021a,b) which highlight different perspectives of the weighted setting.

Open Problem 5. *What are the best possible approximations for these weighted fairness notions? Do weighted EF1 allocations exist for chores?*

6.3 With Monetary Transfers

Since fairness notion like envy-freeness cannot be satisfied exactly, there are works studying how to use payments or subsidies to compensate agents and achieve fairness accordingly (Halpern and Shah, 2019; Brustle et al., 2020). The problem has been extensively considered in the economics literature under the context of rent division problems (Edward Su, 1999). Halpern and Shah (2019) aimed at bounding the amount of external subsidies when the marginal value of each item is at most one for every agent, and Brustle et al. (2020) proved that at most one unit of subsidies per agent is sufficient to guarantee the existence of an envy-free allocation. Caragiannis and Ioannidis (2021) studied the optimization problem of computing allocations that are what they term envy-freeable using the minimum amount of subsidies, and designed a fully polynomial time approximation scheme for instances with a constant number of agents. A more general problem is the fair allocation of mixture of divisible and indivisible items, where the divisible item can be viewed as heterogeneous subsidies (Bei et al., 2021a,b).

6.4 Partial Information

Researchers also care about fair allocation with partial information, and particularly the ordinal preference setting, where the algorithm only knows each agent's ranking over all items without the cardinal values. For goods, the best possible approximation ratio of MMS allocations using only ordinal preferences is $\Omega(\log n)$ by Amanatidis et al. (2016) and Halpern and Shah (2021); for chores, constant upper and lower bounds are proved by Aziz et al. (2020a). Recently, Hosseini et al. (2021) proposed the ordinal MMS fairness, which is more robust to cardinal values. Another interesting question is to investigate the query complexity of unknown valuations. In this model the algorithms can access the valuations by making queries to an oracle. Oh et al. (2021) proved that $\Theta(\log m)$ queries suffices to define an algorithm that returns EF1 allocations. In general, it is an important research direction to explore how much knowledge is sufficient to design a fair allocation algorithm.

Open Problem 6. *Explore the trade-off between the amount of knowledge an algorithm has and the fairness guarantee it ensures.*

6.5 General Valuations

Besides additive valuations, we may have more complex and combinatorial preferences that involve substitutabilities and complementarities in the items, including submodular, XOS, and subadditive valuations. Formal definitions and discussions of these valuations can be found in (Nisan, 2000). Some of the results we have discussed in previous sections also apply to general valuations. For example, the envy-cycle elimination algorithm returns an EF1 allocation for monotone combinatorial valuations (Lipton et al., 2004). Plaut and Roughgarden (2020) proved the existence of 0.5-EFX allocations for subadditive valuations. Regarding MMS, Barman and Krishnamurthy (2020) and Ghodsi et al. (2018) designed polynomial time algorithms to compute constant-approximate allocations for submodular and XOS valuations, and $O(\log n)$-approximate for subadditive valuations. Chaudhury et al. (2021c) further designed algorithms to compute allocations that are approximately EFX and simultaneously achieve $O(n)$-approximation to the maximum Nash welfare for subadditive valuations.

Open Problem 7. *Can the approximation ratios regarding EFX and MMS for subadditive valuations be improved?*

In addition to the above settings, there are more in the literature, such as *constrained resources* (Suksompong, 2021), *public resources* (Conitzer et al., 2017; Fain et al., 2018), *group fairness* (Suksompong, 2018; Conitzer et al., 2019), and *dynamic settings* (Aleksandrov and Walsh, 2020). We refer the readers to the mentioned works/surveys and the references therein for more details.

7. BEYOND FAIRNESS: EFFICIENCY AND INCENTIVES

Beyond achieving fairness alone, more and more attention is paid to investigating the extent to which we can design algorithms to compute allocations that are fair and simultaneously satisfy other properties, such as efficiency and truthfulness.

7.1 Computing Fair and Efficient Allocations

Although finding an allocation that maximizes the utilitarian welfare is straightforward (by allocating each item to the agent who has highest value), finding such an allocation within fair allocations is NP-hard (Barman et al., 2019). One interesting research question here is to bound the utilitarian welfare loss by enforcing the allocations to be fair, i.e., the *price of fairness* (Bei et al., 2021c; Barman et al., 2020). Besides utilitarian welfare, a large body of works studied the compatibility between fairness notions and the weaker efficiency notion of PO. For the case of goods, Caragiannis et al. (2019b) proved that the allocation that maximizes Nash welfare is EF1 and PO. Later, Barman et al. (2018b) designed a pseudopolynomial time algorithm for computing EF1+PO allocations. Truly polynomial time algorithms for the problem remain unknown. Barman and Krishnamurthy (2019) designed a polynomial time algorithm for computing PROP1+PO allocations. Regarding the stronger fairness notion of EFX, Amanatidis et al. (2021a) proved that for bi-valued valuations, the allocation that maximizes Nash welfare is EFX+PO, and Garg and Murhekar (2021) improved this result by giving a polynomial time algorithm. Further, Garg and Murhekar (2021) proved that if the valuations have three different values, EFX+PO allocations may not exist. In contrast, Hosseini et al. (2021) proved that if the valuations are lexicographic, EFX+PO allocations exist and can be found in polynomial time.

Open Problem 8. *Can EF1+PO allocations be computed in truly polynomial time?*

For chores, most of the problems are still open. The good news is that PROP1+PO allocations can be computed in polynomial time, even if the items are a mixture of goods and chores (Aziz et al., 2020b). However, for EF1 and PROPX, their compatibility with PO are still unknown. Ebadian et al. (2021) and Garg et al. (2021b) proved that for bi-valued instances, EF1+PO allocations always exist and can be found efficiently, which are the only exceptions so far.

Open Problem 9. *Do EF1/PROPX + PO allocations always exist for chores?*

7.2 Being Fair for Strategic Agents

Fair allocation problems are often faced by *strategic* agents in real-life scenarios, where an agent may intentionally misreport her values for the items to manipulate the outcome of the algorithm and obtain a bundle of higher value. The goal is to design *truthful* algorithms in which agents maximize their utilities by reporting true preferences. For two agents, Amanatidis et al. (2017a) gave a complete characterization of truthful algorithms, using which we have the tight approximation bounds for solution concepts such as EF1 and MMS. For an arbitrary number of agents, Amanatidis et al. (2016) and Aziz et al. (2019b) designed truthful approximation algorithms but the tight bounds are still unknown. The aforementioned works consider the social environment where monetary transfers are not allowed. With monetary transfers, polynomial time truthful mechanisms were designed by Barman et al. (2019) for single-parameter valuations, which maximize the social welfare and approximately satisfy fairness notions such as MMS and EF1.

Open Problem 10. *What are the best possible approximation ratios of EF1 and MMS for truthful algorithms with an arbitrary number of agents?*

Another game-theoretic research agenda is to investigate the agents' strategic behaviours in algorithms that may not be truthful. For example, Amanatidis et al. (2021b) proved that in the Round-Robin algorithm, the allocations induced by pure Nash equilibria are always EF1 (regarding the true values). Bouveret and Lang (2014) and Aziz et al. (2017a) studied the strategic setting in general sequential allocation algorithms. It is interesting to study agents' behaviours in other algorithms and with other fairness notions.

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